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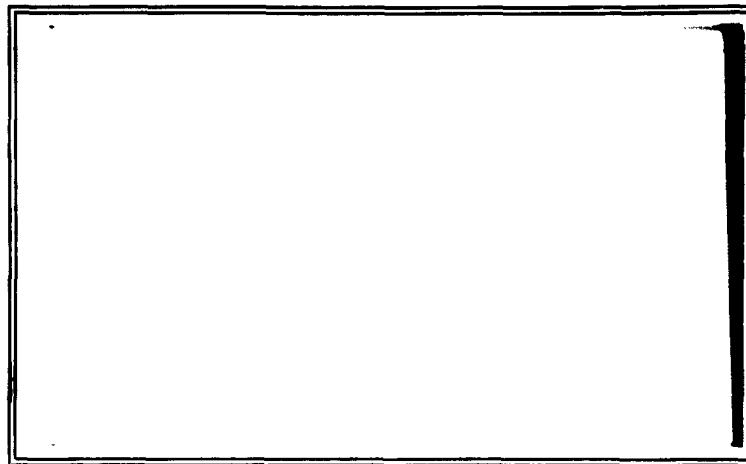
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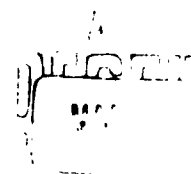
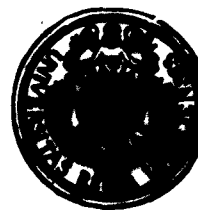
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TECHNICAL NOTE ~~4~~3

Computation of the Emissivity  
of a Cylindrically Symmetric Light Source  
from Measurements of the Projected  
Intensity Profile

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July 31, 1961

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COMPUTATION OF THE EMISSIVITY OF A CYLINDRICALLY SYMMETRIC LIGHT  
SOURCE FROM MEASUREMENTS OF THE PROJECTED INTENSITY PROFILE

S.I. Herlitz

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A method is described for computing the emissivity  $f(r)$  ( $r \leq 1$ ) of a cylindrically symmetric, optically thin light source, when the projected intensity profile  $I(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(r) dy$  is known from experiment. The unknown function  $f(r)$  is taken as a series expansion in terms of orthogonal polynomials, and it is shown how the expansion coefficients can be determined from  $I(x)$ . This procedure yields a least-squares smoothed approximation for  $f(r)$ .

Let  $f(r)$  be the emission coefficient (emission per unit area) of a cylindrically symmetric, optically thin light source of unit radius. Side-on measurements of the emission give the projected intensity profile  $I(x)$ , which is related to  $f(r)$  by

$$I(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(r) dy = 2 \int_x^1 \frac{f(r) r dr}{(r^2 - x^2)^{3/2}} \quad (1)$$

The inversion of eq. (1), giving  $f(r)$  in terms of  $I(x)$ , is

$$f(r) = -\frac{1}{\pi} \int_r^1 \frac{I(x) dx}{(x^2 - r^2)^{3/2}} = -\frac{2}{\pi} \frac{1}{r^2} \int_r^1 \frac{I(x) x dx}{(x^2 - r^2)^{3/2}} \quad (2)$$

The problem of determining  $f(r)$  when  $I(x)$  is known from experiment has been treated by several authors,<sup>1-3</sup> mostly by some method of strip integration. An accurate numerical method has been given by Bockasten.<sup>2</sup>

The x-axis is divided in a number of equal intervals, and a third-degree polynomial is fitted to the  $I(x)$  curve within each interval. Eq. (2) is used, and a matrix  $A_{ik}$  is obtained that transforms a set of values of  $I$  to values of  $f$ ,

$$f(r_i) = \sum_k A_{ik} I(x_k). \quad (3)$$

Bockasten also discusses the influence of random errors in the values of  $I(x)$ .

The method described in this note should be advantageous whenever the  $I(x)$  curve has a more or less irregular shape, and a properly smoothed approximation to  $f(r)$  is desired. The unknown function  $f(r)$  is expanded in a series of orthogonal polynomials. The expansion coefficients can be determined from  $I(x)$ , and in this way a least-squares smoothed approximation to  $f(r)$  can be found directly.

A few general consequences of eqs. (1) and (2) are of interest. Obviously  $I(x)$  is an even function, and we consider the interval  $0 \leq x \leq 1$  only. The behavior of  $f(r)$  near  $r=1$  is very sensitive to the behavior of  $I(x)$  near  $x=1$ . If  $I(x)$  is proportional to  $(1-x)^b$  near  $x=1$ , then  $f(r)$  will be proportional to  $(1-r)^{b-\frac{1}{2}}$  near  $r=1$ . For instance,  $b = 0.49$ ,  $b = 0.5$ , and  $b = 0.51$  correspond to infinite, finite, and zero values of  $f(1)$ , respectively. Further, it is seen from eq. (2) that a discontinuity in  $I'(x)$  at  $x=0$  implies an infinite value of  $f(0)$ . We will assume that the functions  $I(x)$  and  $f(r)$  are finite, so that  $I'(0) = 0$ , and  $I(1) = 0$  (with  $b \geq \frac{1}{2}$ ).

We now choose to fit an even polynomial  $T_n(r)$ , of degree  $2n$ , to  $f(r)$  in such a way that the error squared, integrated over the cross-section  $r \leq 1$ , is a minimum. Thus,

$$\int_0^1 (f - f_M)^2 r dr = \min., M \text{ fixed.} \quad (4)$$

The orthogonal polynomials suitable for this purpose are the Legendre polynomials  $P_m(t)$ , with  $t = 2r^2 - 1$ . The first few of these are  $P_0 = 1$ ,  $P_1 = 2r^2 - 1$ ,  $P_2 = 6r^4 - 6r^2 + 1$ ,  $P_3 = 20r^6 - 30r^4 + 12r^2 - 1$ .

They satisfy the orthogonality relation

$$2(2m+1) \int_0^1 P_n(2r^2-1) P_m(2r^2-1) r dr = \delta_{mn}, \quad (5)$$

so that the series expansion of  $f(r)$  is

$$f(r) = \sum_{m=0}^{\infty} a_m P_m(2r^2-1) \quad (6)$$

with

$$a_m = 2(2m+1) \int_0^1 f(r) P_m(2r^2-1) r dr. \quad (7)$$

If the series (6) is terminated at  $m=M$ , the result is a  $2M$ -degree polynomial  $f_M$ , which satisfies the condition (4).

We substitute the series (6) in eq. (1), integrate term by term, and obtain<sup>x)</sup>

$$I(x) = \sum_{m=0}^{\infty} 2(2m+1)^{-1} a_m \sin[(2m+1)\cos^{-1}x], \quad (8)$$

or, putting  $x = \cos\theta$ ,

$$I(\cos\theta) = \sum_{m=0}^{\infty} 2(2m+1)^{-1} a_m \sin(2m+1)\theta. \quad (9)$$

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<sup>x)</sup> The functions  $U_n(x) = \sin(n \cos^{-1}x)$  are related to the Chebyshev polynomials  $T_n(x) = \cos(n \cos^{-1}x)$ . They satisfy the orthogonality relation  $\int_{-1}^1 U_k U_n (1-x^2)^{-1/2} dx = 0$ ,  $k \neq n$ , and vanish at the points  $x = \pm 1$ .  $U_n(x)$  is an even function if  $n$  is odd.<sup>4</sup>

It can be shown from eqs. (1) and (7) that

$$2(2m+1)^{-1} a_m = (4/\pi) \int_0^{\pi/2} I(\cos\theta) \sin(2m+1)\theta \, d\theta, \quad (10)$$

so that eq. (1) (with  $x = \cos\theta$ ) transforms the Legendre series of  $f(x)$  into the Fourier series of  $I(\cos\theta)$  (odd sine terms only). It may be noted that the total emission from the light source is

$$2\pi \int_0^1 f(r) r \, dr = 2 \int_0^1 I(x) \, dx = \pi a_0. \quad (11)$$

It is also seen that, if  $f_M(r)$  satisfies the condition (4), then the corresponding function  $I_M(x)$  (the series (8) terminated at  $m=M$ ) will be the "best" approximation for  $I(x)$ , in the sense that the integral

$$\int_0^1 (I - I_M)^2 (1 - x^2)^{-\frac{1}{2}} \, dx$$

is minimized.

The standard methods of Fourier analysis<sup>5</sup> can now be used for determining the coefficients  $a_m$ . One possible procedure is as follows: The series (9) is terminated at some  $m=N$ :

$$I_N(\cos\theta) = \sum_{m=0}^N 2(2m+1)^{-1} a_m \sin(2m+1)\theta, \quad (12)$$

and  $I_N$  is required to coincide with  $I$  in those points in the interval  $0 < \theta < \pi/2$  where the first neglected term ( $\sin(2N+3)\theta$ ) vanishes. These points are

$$\theta = k\pi/(2N+3), \quad k=1, 2, \dots, N+1.$$

and the result is



$$a_m = 2 \frac{2m+1}{2N+3} \sum_{k=1}^{N+1} I(\cos \frac{k\pi}{2N+3}) \sin \frac{(2m+1)k\pi}{2N+3}, \quad m=0, 1, \dots, N. \quad (13)$$

This result can also be obtained by evaluating the integral (10) by the trapezoidal rule. Eq. (13) is exact if  $f(r)$  is an even polynomial of degree  $2(N+1)$  or less, so that the series (6), (8), and (9) contain no terms beyond  $m = N+1$ .

With the  $a_m$  given by eq. (13), the  $2M$ -degree polynomial approximation for  $f(r)$  becomes

$$f_M(r) = \sum_{m=0}^M a_m P_m(2r^2-1) = \sum_{k=1}^{N+1} A_k(r) I(\cos \frac{k\pi}{2N+3}) \quad (14)$$

where

$$A_k(r) = \frac{2}{2N+3} \sum_{m=0}^M (2m+1) \sin \frac{(2m+1)k\pi}{2N+3} P_m(2r^2-1). \quad (M \leq N) \quad (15)$$

As an example we consider the function

$$I = (1 - x^2)^2 = \sin^4 \theta, \quad (16)$$

corresponding to

$$f(r) = (8/3\pi)(1-r^2)^{3/2}. \quad (17)$$

The coefficients  $a_m$ , computed from eq. (13), are listed in Table 1. The exact  $a_m$ , given by eq. (7) or (10) are entered in the last line. They are

$$a_m = \frac{48}{\pi} \frac{1}{(2m-3)(2m-1)(2m+3)(2m+5)}, \quad (18)$$

showing fairly rapid convergence. The convergence is slowest near the points  $r=0$  and  $r=1$ , where  $|P_m| = 1$ . A seven-point analysis ( $N=6$ ) is sufficient to reduce the error in  $f(r)$  to about 0.0006 for  $r=0$  and 0.001 for  $r=1$ .

Table 2 shows the same for

$$I = 1 - x^2 = \sin^2 \theta, \quad (19)$$

corresponding to

$$f(r) = (2/\pi) (1-r^2)^{\frac{1}{2}}. \quad (20)$$

The exact coefficients  $a_m$  are

$$a_m = -\frac{4}{\pi} \frac{1}{(2m-1)(2m+3)}, \quad (21)$$

so that the convergence is fairly slow. Clearly, the behavior of the function (20) near  $r=1$  cannot be represented accurately by a low-degree polynomial. However, a seven-point analysis ( $N=6$ ) suffices to give an error less than about 0.002 for  $r \leq 0.98$ .

Experimental measurements will commonly yield  $I(x)$  curves with irregular fluctuations, rather than smooth functions of the type (16) or (19). In such cases it should be an advantage of this method that, with a suitable choice of  $N$  and  $M$ , a properly smoothed  $f(r)$  is obtained directly.

The influence of random errors in the values of  $I(x)$  can be found as follows. If the error (standard deviation) of  $I$  at each point is  $\Delta I$ , eq. (13) gives for the error  $\Delta a_m$  of  $a_m$ ,

$$(\Delta a_m)^2 = 4 \left( \frac{2m+1}{2N+3} \right)^2 (\Delta I)^2 \sum_{k=1}^{N+1} \sin^2 \frac{(2m+1)k\pi}{2N+3} = \frac{(2m+1)^2}{2N+3} (\Delta I)^2,$$

or

$$\Delta a_m = \frac{2m+1}{(2N+3)^{\frac{1}{2}}} \Delta I. \quad (22)$$

For the function (19), assuming  $\Delta I = 0.002$  and  $N = 6$ , we see from eq. (22) and Table 2 that the error in  $a_5$  is comparable to  $a_5$  itself, so that this source of error is more important than the fairly slow convergence of the Legendre series.

From eq. (14), the error in  $f_j$  is given by

$$(\Delta f_M)^2 = (\Delta I)^2 \sum_{k=1}^{N+1} [A_k(r)]^2, \quad (23)$$

which cannot be expressed in a simple form. We can obtain an estimate for  $\Delta f_M$  from the expression

$$(\Delta f_M)^2 = \sum_{m=0}^M (\Delta a_m)^2 P_m^2, \quad (24)$$

which would be correct if the  $\Delta a_m$  were uncorrelated.  $\Delta f_M$  will be largest at the points  $r=0$  and  $r=1$ , where  $|P_m| = 1$ . We find from eqs. (22) and (24),

$$\Delta f_M = \Delta I \sqrt{\frac{(M+1)(2M+1)(2M+3)}{3(2N+3)}}, \quad r=0, 1. \quad (25)$$

For intermediate values of  $r$ , the error will be smaller, although no simple formula can be given. For instance, for  $r^2 = 0.5$  and  $M = 6$ , eqs. (22) and (24) yield  $\Delta f_6 = 5.9 \times (2N+3)^{-\frac{1}{2}} \Delta I$ , while the end-point value given by eq. (25) is  $21 \times (2N+3)^{-\frac{1}{2}} \Delta I$ . For  $M = 10$  the factor is 9.0 and 42, respectively.

#### Acknowledgment

The author is indebted to Dr. K. Bockasten for valuable discussions and comments.

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4. See, for example, E. Madolung, Die Mathematischen Hilfsmittel des Physikers, 5th ed. (Springer, 1953)
5. See, for example, E. Whittaker and G. Robinson, The Calculus of Observations, 4th ed. (Blackie & Son, 1944)

Table 1

Expansion coefficients  $a_m$  of the function  $(8/3\pi)(1 - r^2)^{3/2}$ , approximate values from eq. (13) with  $N = 0, 1, 3, 6$ , and (last line) correct values from eq. (18).

N	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
0	0.32476						
1	0.33931	-0.44089					
3	0.33957	-0.43660	0.08062	0.00951			
6	0.33953	-0.43657	0.08083	0.01027	0.00306	0.00117	0.00046
( $\infty$ )	0.33953	-0.43654	0.08084	0.01029	0.00305	0.00124	0.00061

Table 2

Expansion coefficients  $a_m$  of the function  $(2/\pi)(1 - r^2)^{1/2}$ , approximate values from eq. (13) with  $N = 0, 1, 3, 6$ , and (last line) correct values from eq. (21).

N	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
0	0.4330						
1	0.4253	-0.2437					
3	0.4245	-0.2538	-0.0580	-0.0216			
6	0.4244	-0.2546	-0.0603	-0.0277	-0.0154	-0.0089	-0.0044
( $\infty$ )	0.4244	-0.2546	-0.0606	-0.0283	-0.0165	-0.0109	-0.0077

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